Stability of Periodic Motion in the Presence of Several Zero, Purely Imaginary Roots, and Roots with Negative Real Parts

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The stability of motion in the critical case of m zero roots, $2n$ purely imaginary roots, and q roots with negative real parts for equations with constant coefficients has been investigated by G. V. Kamenkov.[1] In this article the same case, but for equations with periodic coefficients, will be considered.

A system of differential equations of perturbed motion of the order $(m + 2n + q)$ is considered:

$$
\begin{aligned}
du_i/dt &= U_i(u_1, \ldots, u_m; x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_q) \\
dx_s/dt &= -\lambda_s y_s + X_s(u_1, \ldots, u_m; x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_q) \\
dy_s/dt &= \lambda_s x_s + Y_s(u_1, \ldots, u_m; x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_q) \quad (1) \\
dz_r/dt &= p_{r1}z_1 + \ldots + p_{rq}z_q + Z_r(u_1, \ldots, u_m; x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_q) \\
&(i = 1, \ldots, m;\ s = 1, \ldots, n;\ r = 1, \ldots, q)
\end{aligned}
$$

In system (1) coefficients p_{rj} are all determined by the expressions

$$p_{rj} = c_{rj} + \epsilon f_{rj}(t) \qquad \left(c_{rj} = \frac{1}{\omega}\int_0^\omega p_{rj}(t)dt\right) \qquad (2)$$

where c_{rj} = constants, f_{rj} = periodic functions of t with the period ω, and ϵ = a parameter. Functions U_i, X_s, Y_s, and Z_r are bounded functions of u_i, x_s, y_s, and z_r. We shall assume that the first term in the expansion of these functions is, at least, of the second order, and that the expansion coefficients are bounded periodic functions of t of the same period as that of the coefficients p_{rj}.

If in system (1) and the expansion terms of functions U_i, X_s, Y_s, and Z_r we replace the periodic coefficients with their values averaged over a single period, we obtain

$$
\begin{aligned}
du_i/dt &= U_i \\
dx_s/dt &= -\lambda_s y_s + X_s \\
dy_s/dt &= \lambda_s x_s + Y_s \qquad (3) \\
dz_r/dt &= c_{r1}z_1 + \ldots + c_{rq}z_q + Z_r \\
&(i = 1, \ldots, m;\ s = 1, \ldots, n;\ r = 1, \ldots, q)
\end{aligned}
$$

After averaging the periodic coefficients, system (3) is written in the previous notation. This system has been investigated by G. V. Kamenkov.[1] The determinant of the system has m zero roots, $2n$ purely imaginary roots, and q roots with negative real parts.

In system (3) variables u_i, x_s, and y_s are called the critical and z_r the noncritical variables.

System (3) has the property that, to a first approximation, the critical variables do not exert an influence on the noncritical variables. Accordingly, it is possible to carry out the investigation and construct Lyapunov and Chetaev functions for the equations of first approximation separately for each group of variables. From equations for the group of variables z_r, we conclude that system (3) is stable with respect to all z_r and thus there is no need to consider these equations further. It is sufficient to examine only the equations for the critical variables u_i, x_s, and y_s, eliminating terms containing the variables z_r from the expansions of U_i, X_s, and Y_s.

Hence, the question of the stability of system (1) reduces to an investigation of the following system of equations:

$$
\begin{aligned}
du_i/dt &= U_i(u_1, \ldots, u_m; x_1, \ldots, x_n; y_1, \ldots, y_n) \\
dx_s/dt &= -\lambda_s y_s + X_s(u_1, \ldots, u_m; x_1, \ldots, x_n; y_1, \ldots, y_n) \qquad (4) \\
dy_s/dt &= \lambda_s x_s + Y_s(u_1, \ldots, u_m; x_1, \ldots, x_n; y_1, \ldots, y_n) \\
&(i = 1, \ldots, m;\ s = 1, \ldots, n)
\end{aligned}
$$

We shall also assume that, in this system, U_i, X_s, and Y_s are holomorphic functions of the u_i, x_s, and y_s variables.

Let us introduce the polar coordinates r_s and θ_s, where

$$
\begin{aligned}
x_1 &= r_1 \cos\theta_1, \ldots, x_n = r_n \cos\theta_n \\
y_1 &= r_1 \sin\theta_1, \ldots, y_n = r_n \sin\theta_n
\end{aligned} \qquad (5)
$$

Then system (4) will assume the form

$$
\begin{aligned}
du_i/dt &= U_i(u_1, \ldots, u_m; r_1 \cos\theta_1, \ldots, r_n \cos\theta_n; r_1 \sin\theta_1, \ldots, r_n \sin\theta_n) \\
dr_1/dt &= X_1 \cos\theta_1 + Y_1 \sin\theta_1 \\
d\theta_1/dt &= \lambda_1 + (1/r_1)(Y_1 \cos\theta_1 - X_1 \sin\theta_1) \qquad (6) \\
dr_n/dt &= X_n \cos\theta_n + Y_n \sin\theta_n \\
d\theta_n/dt &= \lambda_n + (1/r_n)(Y_n \cos\theta_n - X_n \sin\theta_n)
\end{aligned}
$$

In system (6) the relations between θ_s and t show that, with respect to u_i and r_s, θ_s may play the same role as t, as Lyapunov has demonstrated in Ref. 2.

By introducing the new variables θ_s instead of t, we can write system (6) in the following form:

$$
\begin{aligned}
\frac{du_i}{d\theta_1} &= \frac{1}{\lambda_1} U_1(u_i,r_s,\theta_s)\left(1 + \frac{1}{\lambda_1 r_1}\Theta_1\right)^{-1} \\
\frac{dr_1}{d\theta_1} &= \frac{1}{\lambda_1} R_1(u_i,r_s,\theta_s)\left(1 + \frac{1}{\lambda_1 r_1}\Theta_1\right)^{-1} \\
\frac{dr_2}{d\theta_2} &= \frac{1}{\lambda_2} R_2(u_i,r_s,\theta_s)\left(1 + \frac{1}{\lambda_2 r_2}\Theta_2\right)^{-1} \qquad (7) \\
\frac{dr_n}{d\theta_n} &= \frac{1}{\lambda_n} R_n(u_i,r_s,\theta_s)\left(1 + \frac{1}{\lambda_n r_n}\Theta_n\right)^{-1}
\end{aligned}
$$

where

$$R_s = X_s \cos\theta_s + Y_s \sin\theta_s \qquad \Theta_s = Y_s \cos\theta_s - X_s \sin\theta_s$$

If we multiply the left-hand sides of the third and subse-

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quent equations of system (7) by, respectively, the expressions

$$\frac{d\theta_2}{d\theta_1} = \frac{\lambda_2}{\lambda_1}\left(1 + \frac{1}{\lambda_2 r_2}\Theta_2\right)\left(1 + \frac{1}{\lambda_1 r_1}\Theta_1\right)^{-1}$$

$$\frac{d\theta_n}{d\theta_1} = \frac{\lambda_n}{\lambda_1}\left(1 + \frac{1}{\lambda_n r_n}\Theta_n\right)\left(1 + \frac{1}{\lambda_1 r_1}\Theta_1\right)^{-1}$$

system (7) is transformed to

$$\frac{du_i}{d\theta_1} = \frac{1}{\lambda_1} U_i \left(1 + \frac{1}{\lambda_1 r_1}\Theta_1\right)^{-1}$$

$$\frac{dr_1}{d\theta_1} = \frac{1}{\lambda_1} R_1 \left(1 + \frac{1}{\lambda_1 r_1}\Theta_1\right)^{-1}$$

$$\frac{dr_n}{d\theta_1} = \frac{1}{\lambda_1} R_n \left(1 + \frac{1}{\lambda_1 r_1}\Theta_1\right)^{-1} \tag{8}$$

In system (8) we then average (over a single period) those coefficients that are functions of $\cos\theta_s$ and $\sin\theta_s$ and which have a period 2π. After this the system assumes the final form

$$\frac{du_i}{d\theta_1} = \frac{1}{\lambda_1}\bar{U}_i + [U_i] \qquad \frac{dr_s}{d\theta_1} = \frac{1}{\lambda_1}\bar{R}_s + [R_s] \tag{9}$$

where the terms of higher orders are included in $[U_i]$ and $[R_s]$.

System (9) describes the case when $(m + n)$ zero roots have $2(m + n)$ groups of solutions.

Thus, the investigation of the stability of the motion described by system (1) with periodic coefficients (2) is reduced to an investigation of the stability of the motion described by system (9) with constant coefficients, when the latter system has $2(m + n)$ groups of solutions for $(m + n)$ zero roots. This case was examined in Ref. 3, and, on the basis of investigations of system (9), it is possible to conclude that G. V. Kamenkov's theorems on the stability or instability of the motion described by system (1) remain in force, if the periodic coefficients (2) of system (1) are bounded functions of t, and if the parameter ϵ does not exceed the value following from Sylvester's conditions.

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Mathematical Simulation of the Dynamics of Certain Processes of Fluidization

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A method is proposed for setting up a pattern for mathematical simulation of the dynamics of certain processes in apparatus with fluidized beds, based on the description of the macrokinetics of heterogeneous processes in such beds, by use of multidimensional phase space. In this type of space, the heterogeneous process is considered a "pseudohomogeneous" one. Some examples illustrating the application of the proposed method are presented.

ONE of the interesting regions in which methods of mathematical simulation are applied is the investigation of the dynamics of different processes in equipment with fluidized beds. For simulation of the dynamics of such processes, one must first of all construct a sufficiently general mathematical model to include within its scope both homogeneous and heterogeneous processes. In many cases it is the heterogeneous processes which determine the transient behavior of equipment.

The present paper is an attempt to solve the problem indicated by assuming that heterogeneous processes in fluidized beds may be considered "pseudohomogeneous" processes in multidimensional phase space. So far as the authors know, the literature on equipment in which fluidized beds exist is concerned only with the dynamics of truly homogeneous processes.[1]

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Phase Space and Zones of Homogeneity

In order possibly to develop an all-inclusive scheme for description of homogeneous and heterogeneous processes, it is convenient to use the idea of phase space in describing the distribution of solid particles. This space is defined in terms of a set of rectangular axes along which we measure different quantities characterizing a particle—for example, the three-space coordinates, the degree of transformation, or any other characteristic of the material, temperature, etc. Each state of the process would correspond in phase space to a particle representing the probability function, and these particles in the aggregate would correspond to the density of a fluid filling all parts of phase space.

Using this description of processes, we shall encounter both real liquids and gases in three-dimensional space and also certain fictitious liquids in multidimensional phase spaces. In this description, the dynamics of a process reduce simply to the motion of the given liquids.

It should be noticed, however, that the states of the liquids in the spaces considered do not determine their motion